SAMPLE UNT DISSERTATION WITH
A TWO LINE TITLE

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CHAPTER 1

IT'S EASY AS 1 2 3

In this chapter root systems and Chevalley bases for specific matrix representations of some of the classical, simple, complex Lie algebras are constructed. Each classical, simple, complex Lie algebra is a Lie subalgebra of $\mathfrak{gl}_m(\mathbb{C})$ for some m. The subalgebra of diagonal matrices in such a Lie algebra will be denoted by H. It turns out that for the matrix representations considered, H is a maximal toral subalgebra.

For positive integers i, j, and n with $1 \leq i, j \leq n$, $e_{i,j}$ denotes the square matrix whose only non-zero entry is a 1 in row i and column j. Denote the $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$ by $\operatorname{diag}(a_1, \ldots, a_n)$. Then

$$\operatorname{diag}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & 0 & & & \ldots & 0 \\ 0 & a_2 & & & \ldots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & \ldots & & a_{n-1} & \\ 0 & & \ldots & & 0 & a_n \end{bmatrix} = \sum_{i=1}^{n} a_i e_{i,i}.$$

Clearly the set $\{ e_{i,i} \mid i \leq i \leq n \}$ is a basis of the vector space of diagonal matrices. Suppose $h = \operatorname{diag}(a_1, \ldots, a_n)$ and $1 \leq i, j \leq n$, then

$$\begin{aligned} [h, e_{i,j}] &= h e_{i,j} - e_{i,j} h \\ &= \sum_{k=1}^{n} a_k \, e_{k,k} e_{i,j} - \sum_{k=1}^{n} a_k \, e_{i,j} e_{k,k} \\ &= \sum_{k=1}^{n} a_k \, \delta_{k,i} e_{k,j} - \sum_{k=1}^{n} a_k \delta_{j,k} e_{i,k} \\ &= a_i \, e_{i,j} - a_j \, e_{i,j} \\ &= (a_i - a_j) \, e_{i,j}. \end{aligned}$$

1.1. Type B_n: Odd dimensional, orthogonal Lie algebras

The odd dimensional, orthogonal Lie algebra $\mathfrak{so}_{2n+1}(\mathbb{C})$, or simply $\mathfrak{so}_{2n+1}$, is the set of all matrices X in $\mathfrak{gl}_{2n}(\mathbb{C})$ such that

$$JX = -X^t J$$

where $J = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{smallmatrix}\right]$. Suppose $X = \left[\begin{smallmatrix} a & s & t \\ u & A & B \\ v & C & D \end{smallmatrix}\right]$ where a is a complex number, s, t, u, v are vectors with n components, and A, B, C, D are $n \times n$ matrices. Then $JX = -X^t J$ if and only if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix} \begin{bmatrix} a & s & t \\ u & A & B \\ v & C & D \end{bmatrix} = \begin{bmatrix} -a & -u^t & -v^t \\ -s^t & -A^t & -C^t \\ -t^t & -B^t & -D^t \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix},$$

which is if and only if

$$a = 0 \quad u = -t^t \quad v = -s^t \quad D = -A^t, \quad B = -B^t, \quad \text{and} \quad C = -C^t.$$

If the i^{th} entry of s is s_i, i^{th} entry of t is t_i, the (i, j) entry of A is $a_{i,j}$, the (i, j) entry of B is $b_{i,j}$, and the (i, j) entry of C is $c_{i,j}$, then $\left[\begin{smallmatrix} a & s & t \\ u & A & B \\ v & C & D \end{smallmatrix}\right]$ is in $\mathfrak{so}_{2n+1}$ if and only if

$$\begin{bmatrix} a & s & t \\ u & A & B \\ v & C & D \end{bmatrix} = \begin{bmatrix} 0 & s_1 & s_2 & \dots & s_n & t_1 & t_2 & \dots & t_n \\ -t_1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} & 0 & b_{1,2} & \dots & b_{1,n} \\ -t_2 & a_{2,1} & a_{2,2} & \dots & a_{2,n} & -b_{2,1} & 0 & \dots & b_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -t_n & a_{n,1} & a_{n,2} & \dots & a_{n,n} & -b_{n,1} & -b_{n,2} & \dots & 0 \\ -s_1 & 0 & c_{1,2} & \dots & c_{1,n} & -a_{1,1} & -a_{n,2} & \dots & -a_{n,1} \\ -s_2 & -c_{2,1} & 0 & \dots & c_{1,n} & -a_{2,1} & -a_{2,2} & \dots & -a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_n & -c_{n,1} & -c_{n,2} & \dots & 0 & -a_{1,n} & -a_{n,2} & \dots & -a_{n,n} \end{bmatrix}.$$

For $1 \leq i \leq n$ define

$$d_i = e_{i+1,i+1} - e_{n+i+1,n+i+1}.$$

Then

$$\mathcal{B}_H = \{\, d_i \mid 1 \le i \le n \,\} = \{\, e_{i+1,i+1} - e_{n+i+1,n+i+1} \mid 1 \le i \le n \,\}.$$

is a basis of H.

For $1 \le i \le n$, define x_i in H^* by

$$x_i(h) = a_i \quad \text{when} \quad h = \operatorname{diag}(0, a_1, \ldots, a_n, -a_1, \ldots, -a_n).$$

Then $x_i(h)$ is the coefficient of d_i when h is expressed as a linear combination of vectors in $\mathcal{B}_H$.

The set

$$\begin{aligned}\mathcal{B} = \mathcal{B}_H \cup \{\, e_{1,j+1} - e_{n+j+1,1} \mid 1 \le j \le n \,\} \cup \{\, e_{1,n+j+1} - e_{j+1,1} \mid 1 \le j \le n \,\} \\ \cup \{\, e_{i+1,j+1} - e_{n+j+1,n+i+1} \mid 1 \le i \ne j \le n \,\} \cup \{\, e_{i+1,n+j+1} - e_{j+1,n+i+1} \mid 1 \le i < j \le n \,\} \\ \cup \{\, e_{n+i+1,j+1} - e_{n+j+1,i+1} \mid 1 \le i < j \le n \,\}\end{aligned}$$

is a basis of $\mathfrak{so}_{2n+1}$. In particular, $\dim \mathfrak{so}_{2n+1} = 3n + n^2 - n + 2\binom{n}{2} = 2n^2 + n$.

Proposition 1.1. *The set $\mathcal{B} \setminus \mathcal{B}_H$ consists of root vectors.*

Proof. This is proved by direct computation. There are five cases.

Suppose that $h = \operatorname{diag}(0, a_1, \ldots, a_n, -a_1, \ldots, -a_n)$ is in H.

Consider $e_{1,j+1} - e_{n+j+1,1}$ where $1 \le j \le n$. Then

$$\begin{aligned}[h, e_{1,j+1} - e_{n+j+1,1}] &= [h, e_{1,j+1}] - [h, e_{n+j+1,1}] \\ &= -a_j\, e_{1,j+1} - a_j\, e_{n+j+1,1} \\ &= (-a_j)\,(e_{1,j+1} - e_{n+j+1,1}) \\ &= (-x_j)(h)\,(e_{1,j+1} - e_{n+j+1,1}).\end{aligned}$$

Thus, $e_{1,j+1} - e_{n+j+1,1}$ is a root vector. The corresponding root is the linear function $-x_j$ in H^*.

Consider $e_{i+1,j+1} - e_{n+j+1,n+i+1}$ where $1 \le i \ne j \le n$. Then

$$\begin{aligned}[h, e_{i+1,j+1} - e_{n+j+1,n+i+1}] &= [h, e_{i+1,j+1}] - [h, e_{n+j+1,n+i+1}] \\ &= (a_i - a_j)\, e_{i+1,j+1} - (-a_j + a_i)\, e_{n+j+1,n+i+1} \\ &= (a_i - a_j)\, (e_{i+1,j+1} - e_{n+j+1,n+i+1}) \\ &= (x_i - x_j)(h)\, (e_{i+1,j+1} - e_{n+j+1,n+i+1}).\end{aligned}$$

Thus, $e_{i+1,j+1} - e_{n+j+1,n+i+1}$ is a root vector. The corresponding root is the linear function $x_i - x_j$ in H^*.

Consider $e_{i+1,n+j+1} - e_{j+1,n+i+1}$ where $1 \le i < j \le n$. Then

$$\begin{aligned}[h, e_{i+1,n+j+1} - e_{j+1,n+i+1}] &= [h, e_{i+1,n+j+1}] - [h, e_{j+1,n+i+1}] \\ &= (a_i + a_j)\, e_{i+1,n+j+1} - (a_j + a_i)\, e_{j+1,n+i+1} \\ &= (a_i + a_j)\, (e_{i+1,n+j+1} - e_{j+1,n+i+1}) \\ &= (x_i + x_j)(h)\, e_{i+1,n+j+1} - e_{j+1,n+i+1}.\end{aligned}$$

Thus, $e_{i+1,n+j+1} - e_{j+1,n+i+1}$ is a root vector. The corresponding root is the linear function $x_i + x_j$ in H^*.

The other two cases are similar: $e_{1,n+j+1} - e_{j+1,1}$ is a root vector and the corresponding root is the linear function x_j in H^*; $e_{n+i+1,j+1} - e_{j+1,n+i+1}$ is a root vector and the corresponding root is the linear function $-x_i - x_j$ in H^*.

The computations above are summarized in Table 1.1. $\square$

Corollary 1.2. *The the subalgebra H is a maximal toral subalgebra and the root system of $(\mathfrak{so}_{2n+1}, H)$ is*

$$\Phi = \{\pm(x_i \pm x_j) \mid 1 \le i < j \le n\} \cup \{\, 2x_i \mid 1 \le i \le n \,\}.$$

Proof. By the proposition, $\mathfrak{so}_{2n+1}$ has a root space decomposition. Suppose that H' is a toral subalgebra containing H. Just suppose that H' properly contains H. Then H' is abelian and there is an element h' in H that is a linear combination of the basis elements in

$i,\ j$	α	e_α
$1 \le j \le n$	$-x_j$	$e_{1,j+1} - e_{n+j+1,1}$
$1 \le j \le n$	x_j	$e_{1,n+j+1} - e_{j+1,1}$
$1 \le i \ne j \le n$	$x_i - x_j$	$e_{i+1,j+1} - e_{n+j+1,n+i+1}$
$1 \le i < j \le n$	$x_i + x_j$	$e_{i+1,n+j+1} - e_{j+1,n+i+1}$
$1 \le i < j \le n$	$-x_i - x_j$	$e_{n+i+1,j+1} - e_{j+1,n+i+1}$

TABLE 1.1. Roots and root vectors for $\mathfrak{so}_{2n+1}(\mathbb{C})$

$\mathcal{B} \setminus \mathcal{B}_H$. Write $h' = v_\alpha + h''$ where v_α is a non-zero vector in the α root space. Then v_α is a non-zero multiple of the root vector e_α in $\mathcal{B} \setminus \mathcal{B}_H$. Fix h in H such that h is not in $\ker \alpha$, then $[h, h'] = [h, v_\alpha + h''] = \alpha(h)v_\alpha + [h, h'']$. Then $\alpha(h)v_\alpha \ne 0$ and $[h, h'']$ is in the span of $\mathcal{B} \setminus (\mathcal{B}_H \cup \{e_\alpha\})$. Therefore, $[h, h'] \ne 0$. This contradicts the fact that H' is abelian. Thus, $H' = H$ and so H is maximal. $\square$

For $1 \le i \le n$ define α_i in H^* by

$$\alpha_i = x_i - x_{i+1} \quad (1 \le i \le n-1),$$
$$\alpha_n = x_n.$$

Set $\Pi = \{\, \alpha_i \mid 1 \le i \le n \,\}$. It's easy to see that Π is a basis of H^*. In Table 1.2 each root in Φ is given as a linear combination of roots in Π. Notice that the roots $x_i - x_j$ with $i \ne j$ from Table 1.1 are split into two subsets depending on whether or not $i < j$.

By direct inspection, there is a unique root with maximal height, this is the *highest root*. The highest root is $x_1 + x_2 = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$ and its height is $2n - 1$.

The usual Euclidean metric on H^* is defined by

$$d(\sum_{i=1}^n a_i x_i, \sum_{i=1}^n b_i x_i) = \sqrt{\sum_{i=1}^n |a_i - b_i|^2}.$$

$i,\ j$	α	$=$	$\sum_{i=1}^n m_i\alpha_i$	$\mathrm{ht}(\alpha)$
$1 \le i < j \le n$	$x_i - x_j$	$=$	$\alpha_i + \cdots + \alpha_{j-1}$	$j - i$
$1 \le i \le n$	x_i	$=$	$\alpha_i + \cdots + \alpha_{n-1} + \alpha_n$	$n - i + 1$
$1 \le i < j \le n$	$x_i + x_j$	$=$	$\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + 2\alpha_n$	$2n - i - j + 2$
$1 \le i < j \le n$	$-x_i + x_j$	$=$	$-\alpha_i - \cdots - \alpha_{j-1}$	$-j + i$
$1 \le i \le n$	$-x_i$	$=$	$-\alpha_i - \cdots - \alpha_{n-1} - \alpha_n$	$-n + i - 1$
$1 \le i < j \le n$	$-x_i - x_j$	$=$	$-\alpha_i - \cdots - \alpha_{j-1} - 2\alpha_j - \cdots - 2\alpha_{n-1} - 2\alpha_n$	$-2n + i + j - 2$

TABLE 1.2. Roots expressed as linear combinations of vectors in Π

With respect to this metric, the roots $\pm(x_i \pm x_j)$ with $i \neq j$ have length $\sqrt{2}$ and the roots $\pm x_i$ have length 1. Thus, there are two root lengths. Roots with minimum length are called *short roots* and roots with maximum length are called *long roots.* The highest root is a long root.

By direct inspection, there is a unique highest short root, $x_1 = \alpha_1 + \cdots + \alpha_n$, with height n.

Notice that if $\alpha = \sum_{i=1}^n m_i\alpha_i$, then the coefficients m_i are either all non-negative or all non-positive. Define

$$\Phi^+ = \left\{ \alpha = \sum_{i=1}^n m_i\alpha_i \mid mi_{\geq}0 \,\forall\, 1 \le i \le n \right\}$$

and

$$\Phi^- = \left\{ \alpha = \sum_{i=1}^n m_i\alpha_i \mid mi_{\leq}0 \,\forall\, 1 \le i \le n \right\}.$$

Then $\Phi^- = -\Phi^+$ and $\Phi = \Phi^+ \coprod \Phi^-$.

We next compute the elements t_{α_i} in H for $1 \le i \le n$. Using the basis $\mathcal{B}$ of $\mathfrak{so}_{2n+1}$ it is straightforward to compute the restriction of the Killing form to H by computing the matrices of $\operatorname{ad} h$ and $\operatorname{ad} h'$, and then $\operatorname{tr}(\operatorname{ad} h \circ \operatorname{ad} h')$ for h and h' in H. The result is

$$\kappa(h, h') = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h').$$

If $h = \operatorname{diag}(0, a_1, \ldots, a_n, -a_1, \ldots, -a_n)$ and $h' = \operatorname{diag}(0, a'_1, \ldots, a'_n, -a'_1, \ldots, -a'_n)$, then $\alpha(h)\alpha(h')$ is given in Table 1.3.

α	$\alpha(h)\alpha(h')$
$x_i - x_j$	$(a_i - a_j)(a'_i - a'_j)$
x_i	$(a_i)(a'_i) = a_i a'_i$
$x_i + x_j$	$(a_i + a_j)(a'_i + a'_j)$
$-x_i + x_j$	$(-a_i + a_j)(-a'_i + a'_j) = (a_i - a_j)(a'_i - a'_j)$
$-x_i$	$(-a_i)(-a'_i) = a_i a'_i$
$-x_i - x_j$	$= (-a_i - a_j)(-a'_i - a'_j) = (a_i + a_j)(a'_i + a'_j)$

TABLE 1.3. $\alpha(h)\alpha(h')$ when $h = \operatorname{diag}(a_1, \ldots, a_n)$ and $h' = \operatorname{diag}(a'_1, \ldots, a'_n)$

We can now compute $\kappa(h, h')$ in terms of the coefficients of h and h' when h and h' are expressed as linear combinations of $\{x_1, \ldots, x_n\}$.

$$\begin{aligned}
\kappa(h, h') &= \sum_{1 \le i < j \le n} 2\left((a_i - a_j)(a'_i - a'_j) + (a_i + a_j)(a'_i + a'_j)\right) + 2\sum_{i=1}^{n}(a_i a'_i) \\
&= \sum_{1 \le i < j \le n} (4a_i a'_i + 4a_j a'_j) + \sum_{i=1}^{n} 2a_i a'_i \\
&= \sum_{i=1}^{n} a_i a'_i (2 + 4(n - i) + 4(i - 1)) \\
&= (4n - 2)\sum_{i=1}^{n} a_i a'_i.
\end{aligned} \tag{1}$$

The penultimate equality in (1) is most easily seen by arranging the summands in an $n \times n$

array.

$$\begin{array}{ccccccc}
2a_1a_1' & 4a_1a_1'+4a_2a_2' & 4a_1a_1'+4a_3a_3' & \dots & \dots & \dots & 4a_1a_1'+4a_na_n' \\
 & 2a_2a_2' & 4a_2a_2'+4a_3a_3' & & & & 4a_2a_2'+4a_na_n' \\
 & & 2a_3a_3' & & & & 4a_3a_3'+4a_na_n' \\
 & & & & & & \vdots \\
 & & & & \ddots & & \vdots \\
 & & & & & & 4a_{n-2}a_{n-2}'+4a_na_n' \\
 & & & & & & 2a_na_n'
\end{array}$$

For $1 \le i \le n$. Then the element t_{α_i} in H is defined by the condition that

$$\kappa(h, t_{\alpha_i}) = \alpha_i(h) \quad \text{for all } h \text{ in } H.$$

Fix $1 \le i \le n-1$ and suppose $t_{\alpha_i} = \mathrm{diag}(0, t_1, \dots, t_n, -t_1, \dots, -t_n)$. Then

$$a_i - a_{i+1} = (4n-2)(a_1t_1 + \cdots + a_it_i + a_{i+1}t_{i+1} + \cdots + a_nt_n)$$

when $h = \mathrm{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n)$. Thus, $t_1, \dots, t_n$ are such that

$$a_1t_1 + \cdots + a_i\big(t_i - \tfrac{1}{4n-2}\big) + a_{i+1}\big(t_{i+1} + \tfrac{1}{4n-2}\big) + \cdots + a_nt_n = 0$$

for all $a_1, \dots, a_n$ in $\mathbb{C}$. Taking $a_j = 1$ and $a_k = 0$ for $k \ne j$ we see that

$$t_j = \begin{cases} \frac{1}{4n-2} & j = i \\ -\frac{1}{4n-2} & j = i+1 \\ 0 & j \ne i, i+1. \end{cases}$$

Therefore, for $1 \le i \le n-1$, $t_{\alpha_i} = \frac{1}{4n-2}(d_i - d_{i+1})$.

Now consider t_{α_n}. Say $t_{\alpha_i} = \mathrm{diag}(0, t_1, \dots, t_n, -t_1, \dots, -t_n)$. Then

$$a_n = (4n-2)(a_1t_1 + \cdots + a_it_i + a_{i+1}t_{i+1} + \cdots + a_nt_n)$$

when $h = \mathrm{diag}(0, a_1, \dots, a_n, -a_1, \dots, -a_n)$. Thus, $t_1, \dots, t_n$ are such that

$$a_1t_1 + \cdots + a_{n-1}t_{n-1} + a_n\big(t_n - \tfrac{1}{4n-2}\big) = 0$$

for all $a_1, \ldots, a_n$ in $\mathbb{C}$. Taking $a_j = 1$ and $a_k = 0$ for $k \neq j$ we see that

$$t_j = \begin{cases} \frac{1}{4n-2} & j = n \\ 0 & j \neq n. \end{cases}$$

Therefore, $t_{\alpha_n} = \frac{1}{4n-2} d_n$.

For $1 \leq i \leq n-1$ we have

$$\kappa(t_{\alpha_i}, t_{\alpha_i}) = (4n-2)\left(\tfrac{1}{(4n-2)^2} + \tfrac{1}{(4n-2)^2}\right) = \frac{1}{2n-1}.$$

Also

$$\kappa(t_{\alpha_n}, t_{\alpha_n}) = (4n-2)\frac{1}{(4n-2)^2} = \frac{1}{4n-2}.$$

Therefore

$$h_{\alpha_i} = (4n-2)t_{\alpha_i} = d_i - d_{i+1} \quad (1 \leq i \leq n-1),$$

$$h_{\alpha_n} = (8n-4)t_{\alpha_n} = 2d_n.$$

The *Cartan matrix* of $\mathfrak{so}_{2n+1}$ (or of Φ, when t_{α_i} is identified with $\check{\alpha}_i$) is the matrix $C(\mathfrak{so}_{2n+1})$ whose (i,j)-entry is $\alpha_i(h_{\alpha_j})$. Using the computations above we see that

$$C(\mathfrak{so}_{2n+1}) = \begin{bmatrix} 2 & -1 & 0 & & & \ldots & 0 \\ -1 & 2 & -1 & & & \ldots & 0 \\ 0 & -1 & 2 & & & \ldots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & & \ldots & & 2 & -1 & 0 \\ 0 & & \ldots & & -1 & 2 & -2 \\ 0 & & \ldots & & 0 & -1 & 2 \end{bmatrix}.$$

1.2. Type D_n: Even dimensional, orthogonal Lie algebras

CHAPTER 2

AS SIMPLE AS DO RE MI

2.1. Definition of $\mathcal{H}$ and the Uniformly Expanding Property

In this section we define the family $\mathcal{H}$ and we establish basic dynamical properties of a map $f_a \in \mathcal{H}$. Then we we prove the important Lemma 2.1.

$i,\ j$	α	$=$	$\sum_{i=1}^{n} m_i \alpha_i$	$\mathrm{ht}(\alpha)$
$1 \le i < j \le n$	$x_i - x_j$	$=$	$\alpha_i + \cdots + \alpha_{j-1}$	$j - i$
$1 \le i \le n$	x_i	$=$	$\alpha_i + \cdots + \alpha_{n-1} + \alpha_n$	$n - i + 1$
$1 \le i < j \le n$	$x_i + x_j$	$=$	$\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + 2\alpha_n$	$2n - i - j + 2$
$1 \le i < j \le n$	$-x_i + x_j$	$=$	$-\alpha_i - \cdots - \alpha_{j-1}$	$-j + i$
$1 \le i \le n$	$-x_i$	$=$	$-\alpha_i - \cdots - \alpha_{n-1} - \alpha_n$	$-n + i - 1$
$1 \le i < j \le n$	$-x_i - x_j$	$=$	$-\alpha_i - \cdots - \alpha_{j-1} - 2\alpha_j - \cdots - 2\alpha_{n-1} - 2\alpha_n$	$-2n + i + j - 2$

TABLE 2.1. Roots expressed as linear combinations of vectors in Π

2.1.1. Definition of $\mathcal{H}$

We define the family $\mathcal{H}$ as a family of maps in the Speiser class of transcendental entire functions of finite singular type.

Let $a = (a_0, a_1, \cdots, a_n) \in \mathbb{C}^{n+1}$ be a vector such that $a_0 \neq 0$, $a_n \neq 0$,

$$P_a(z) = a_n z^n + \cdots + a_1 z + a_0 \in \mathbb{C}[z]$$

and

$$g_a(z) = \frac{P_a(z)}{z^k}$$

where k is a positive integer strictly less than $n = \deg(P_a) \geq 2$. Define

$$f_a(z) = g_a \circ \exp(z) = \tfrac{a_n e^{nz} + a_{n-1} e^{(n-1)z} + \cdots + a_1 e^z + a_0}{e^{zk}} = \textstyle\sum_{j=0}^{n} a_j e^{(j-k)z}$$

Observe that maps of this form do not have any finite asymptotic values. This is the reason why we restricted ourselves to integers k satisfying condition $0 < k < n$. As it was mentioned in Chapter 1, the most well known examples of this type of maps are maps from the *cosine* family.

We denote by $\text{Crit}(f_a)$ the set $\{z : f_a'(z) = 0\}$. Observe that

$$f_a'(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}$$

and that $g_a'(z) = 0$ if and only if $zP_a'(z) - kP_a(z) = 0$, which is equivalent to

$$\sum_{j=0}^{n} a_j (j-k) z^j = 0.$$

Therefore, there exist n non-zero complex numbers (counting multiplicities) $s_1, s_2, \cdots, s_n$ such that $z \in \text{Crit}(f_a)$ if and only if $e^z = s_k$ for some $k = 1, 2, \cdots, n$ i.e.

$$\{z_k = \log s_k + 2\pi i m : m \in \mathbb{Z}, k = 1, \cdots, n\}$$

is the set of critical points and observe that the set of critical values of a map f_a is finite.

Denote by $\mathcal{H}$ the family of functions

$$\mathcal{H} = \left\{ f_a(z) = \frac{P_a(e^z)}{e^{kz}} : \deg P_a > k > 0 \text{ and } \delta_a > 0 \right\},$$

where by $\mathcal{P}_{f_a}$ we denote the post-critical set of f_a, that is, the set

$$\mathcal{P}_{f_a} = \overline{\bigcup_{n \geq 0} f_a^n(Crit(f_a))}$$

and

$$\delta_a = \frac{1}{2} \min \left\{ \frac{1}{2}, \text{dist}(J_{f_a}, \mathcal{P}_{f_a}) \right\},$$

where

$$\text{dist}(J_{f_a}, \mathcal{P}_{f_a}) = \inf\{|z_1 - z_2| : z_1 \in J_{f_a}, z_2 \in \mathcal{P}_{f_a}\}$$

is the Euclidean distance between the Julia set of f_a, J_{f_a}, and the post-critical set of f_a, $\mathcal{P}_{f_a}$.

The reason we define δ_a in such a way will be more visible later on, starting with Chapter 3, and is due to the application (we shall need) of the Koebe Distortion Theorem since one can observe that, for every $y \in J_{f_a}$ and for every $n \geq 1$, there exists a unique holomorphic inverse branch

$$(f_a^n)_y^{-1} : B(f_a^n(y), 2\delta_a) \to \mathbb{C}$$

such that $(f_a^n)_y^{-1} \circ (f_a^n)(y) = y$.

Then there exists a numerical constant K such that, for $z_1, z_2 \in J_{f_a}$ with $|z_1 - z_2| < \delta_a$ and for $y \in f_a^{-n}(z_1)$,

$$\frac{1}{K} \leq \frac{|((f_a^n)_y^{-1})'(z_1)|}{|((f_a^n)_y^{-1})'(z_2)|} \leq K. \tag{2}$$

Observe that $Crit(f_a) \subset F_{f_a}$, where F_{f_a} is the Fatou set of f_a. Consequently, maps in the family $\mathcal{H}$ do not have neither parabolic domains nor Herman rings nor Siegel disks. Moreover, as was written in Chapter 1 they do not have neither wandering nor Baker domains. Also for every point z in the Fatou set there exists (super)attracting cycle such that the trajectory of z converges to this cycle.

2.1.2. The Cylinder and the Definition of $J_{F_a}^r$

Since the map $f_a \in \mathcal{H}$ is periodic with period $2\pi i$, we consider it on the quotient space $P = \mathbb{C}/\sim$ (the cylinder) where

$$z_1 \sim z_2 \text{ iff } z_1 - z_2 = 2k\pi i \text{ for some } k \in \mathbb{Z}.$$

If $\pi\colon \mathbb{C} \to P$ is the natural projection, then, since the map $\pi \circ f_a : \mathbb{C} \to P$ is constant on equivalence classes of relation $\sim$, it induces a holomorphic map

$$F_a : P \to P.$$

The cylinder P is endowed with Euclidean metric which will be denoted in what follows by the same symbol $|w - z|$ for all $z, w \in P$. The Julia set of F_a is defined to be

$$J_{F_a} = \pi(J_{f_a})$$

and observe that

$$F_a(J_{F_a}) = J_{F_a} = F_a^{-1}(J_{F_a}).$$

We shall study the set $J^r_{f_a}$ consisting of those points of J_{f_a} that do not escape to infinity under positive iterates of f_a. In other words, if

$$I_\infty(f_a) = \{z \in \mathbb{C} : \lim_{n\to\infty} f_a^n(z) = \infty\},$$

then

$$J^r_{f_a} = J_{f_a} \backslash I_\infty(f_a)$$

and, if

$$I_\infty(F_a) = \{z \in P : \lim_{n\to\infty} F^n(z) = \infty\},$$

then

$$J^r_{F_a} = J_{F_a} \backslash I_\infty(F_a).$$

In what follows we fix $a \in \mathbb{C}^{n+1}$ and we denote for simplicity $f_a \in \mathcal{H}$ by f. The following Lemma reveals some background information for a better understanding of the dynamical behavior of maps in our family $\mathcal{H}$. This lemma will be used several times and it will be a key technical ingredient for many proofs.

Observe first that, if we consider $a = (a_0, \cdots, a_n) \in \mathbb{C}^{n+1}$, since

$$f_a(z) = \sum_{j=0}^{n} a_j e^{(j-k)z} \tag{3}$$

we have

$$f'_a(z) = \sum_{j=0}^{n} a_j (j-k) e^{(j-k)z}. \tag{4}$$

Lemma 2.1. *Let f_a be a function of form (3). Then there exist $M_1, M_2, M_3 > 0$ such that, for every z with $|Re\ z| \geq M_3$, the following inequalities hold.*

(1) $M_1 e^{q|Re\ z|} \leq |f_a(z)| \leq M_2 e^{q|Re\ z|}$

(2) $M_1 e^{q|Re\ z|} \leq |f'_a(z)| \leq M_2 e^{q|Re\ z|}$

(3) $\frac{M_1}{M_2}|f'_a(z)| \leq |f_a(z)| \leq \frac{M_2}{M_1}|f'_a(z)|$

where $q = \begin{cases} k & \text{if } Re\ z < 0 \\ n-k & \text{if } Re\ z > 0. \end{cases}$

PROOF. Note that (iii) follows from (i) and (ii). The proof of (i) and (ii) follows from the fact that

$$|f_a(z)| = |a_n|e^{(n-k)\mathrm{Re}\ z} + o(e^{(n-k)\mathrm{Re}\ z}) \text{ as Re } z \to \infty$$

$$|f_a(z)| = |a_0|e^{-k\mathrm{Re}\ z} + o(e^{-k\mathrm{Re}\ z}) \text{ as Re } z \to -\infty$$

and from the observation that f_a' is a function of the same (algebraic) type as f_a (see (4)). $\square$

2.1.3. The Uniformly Expanding Property

In this section we shall prove, mainly, the very important result, Proposition 2.2, using McMullen's result from [1], that any map $f_a \in \mathcal{H}$ is uniformly expanding on its Julia set.

PROPOSITION 2.2. *For every $f \in \mathcal{H}$ there exist $c > 0$ and $\gamma > 1$ such that*

$$|(f^n)'(z)| > c\gamma^n$$

for every $z \in J_f$.

PROOF. By [1, Proposition 6.1], for all $z \in J_f$,

$$\lim_{n\to\infty} |(f^n)'(z)| = \infty. \tag{5}$$

Since f is periodic with period $2\pi i$ we consider

$$A = J_f \cap \{z : \mathrm{Im}\ z \in [0, 2\pi]\}$$

and we let A_m denotes the open set

$$\{z \in A : |(f^m)'(z)| > 2\}.$$

Then by (5) $\{A_m\}_{m\geq 1}$ is an open covering of A. Moreover, it follows from Lemma 2.1 that there exists M such that, if $|\mathrm{Re}\ z| > M$, then $|f'(z)| > 2$. Therefore

$$\{z \in A : |\mathrm{Re}\ z| > M\} \subset A_1.$$

Since $A \cap \{z : |\mathrm{Re}\ z| \leq M\}$ is a compact subset of A, it follows that there exists $k \geq 1$ such that the family $\{A_1, A_2, \ldots, A_k\}$ covers A. It implies that, for every $z \in A$, there exists

$k(z) \leq k$ for which $|(f^{k(z)})'(z)| > 2$. Therefore, for every $n > 0$ and every $z \in A$ we can split the trajectory $z, f(z), \ldots, f^n(z)$ into $l \leq \lfloor \frac{n}{k} \rfloor + 1$ pieces of the form

$$z_i, f(z_i), \ldots, f^{k(z_i)-1}(z_i)$$

for $i = 1, \ldots, l-1$, and, for $i = l$,

$$z_l, f(z_l), \ldots f^j(z_l) = f^n(z),$$

where $z_1 = z$, $z_i = f^{k(z_{i-1})}(z_{i-1})$ and j is some integer smaller than k. Then

$$|(f^n)'(z)| \geq 2^{\lfloor \frac{n}{k} \rfloor} \Delta^{k-1},$$

where

$$\Delta = \inf_{z \in J_f} |f'(z)| \neq 0,$$

since J_f contains no critical points and because of Lemma 2.1 (ii). It follows that

$$|(f^n)'(z)| \geq 2^{\frac{n}{k}-1} \Delta^{k-1} = \frac{\Delta^{k-1}}{2} (2^{\frac{1}{k}})^n.$$

$\square$

2.2. Bounded Orbits and Classical Conformal Repellers.

We fix again $a \in \mathbb{C}^{n+1}$ and we denote f_a by f, F_a by F and the Julia set of F by J_F. Our goal in this section is to prove Proposition 2.5. In order to prove this proposition we apply the thermodynamic formalism for compact repellers.

Definition 2.3. Let f be a holomorphic function from an open subset V of $\mathbb{C}$ into $\mathbb{C}$ and J a compact subset of V. The triplet (J, V, f) is a conformal repeller if

(1) there are $C > 0$ and $\alpha > 1$ such that $|(f^n)'(z)| \geq C\alpha^n$ for every $z \in J$ and $n \geq 1$.

(2) $f^{-1}(V)$ is relatively compact in V with

$$J = \bigcap_{n \geq 1} f^{-n}(V).$$

(3) for any open set U with $U \cap J$ not empty, there is $n > 0$ such that

$$J \subset f^n(U \cap J).$$

It is worth noting that there are no critical points of f in J.

2.2.1. Conformal Repellers

Let (J, V, g) be a (mixing) conformal expanding repeller(see for example [2] for more properties). In the proof of Proposition 2.5, $J = J_1(M)$ is a compact subset of $\mathbb{C}$, limit of a finite conformal iterated function system, $g = F$, is a holomorphic function for which J is invariant and for which there exist $\gamma > 1$ and $c > 0$ such that, for all $n \in \mathbb{N}$ and for all $z \in J$, $|(g^n)'(z)| \geq c\gamma^n$. For $t \in \mathbb{R}$ we consider the topological pressure defined by

$$P_z(t) = \lim_{n\to\infty} \frac{1}{n} \log P_z(n, t),$$

where

$$P_z(n, t) = \sum_{y \in g^{-n}(z)} |(g^n)'(y)|^{-t}.$$

The function $P(t) = P_z(t)$ as a function of t is independent of z, continuous, strictly decreasing, $\lim_{t\to-\infty} P(t) = +\infty$ and the following remarkable theorem holds.

Theorem 2.4 (Bowen's Formula). *Hausdorff dimension of J is the unique zero of $P(t)$.*

For more details and definitions concerning the thermodynamic formalism of conformal expanding repellers (initiated by Bowen and Ruelle) we refer the reader to [2].

In order to prove Proposition 2.5, i.e. to show that $\mathrm{HD}(J) > 1$, we use Bowen's formula and we observe that, from the definition of $P_z(n, t)$, it is enough to find a constant $C > 1$ such that, for all $z \in J$,

$$P_z(1, 1) \geq C. \tag{6}$$

Proposition 2.5. *Let $f \in \mathcal{H}$. Then the Hausdorff dimension of the set of points in Julia set of f having bounded orbit is strictly greater than* 1.

Proof. Let N be a large number, $H = \{z \in \mathbb{C} : \mathrm{Re}\ z > N\}$. Observe that there exists U such that $\overline{U} \subset \{z : s - \pi < \mathrm{Im}\ z < s + \pi\}$ for some $s \in (-\pi, \pi]$, $\mathrm{Re}\ U > 0$, $f|_U$ is univalent and $f(U) = H$. Note that, since N is large, by Lemma 2.1 there exists $\gamma_N > 1$ such that, if Re $z \geq N$, then

$$|F'(z)| = |f'(z)| > \gamma_N. \tag{7}$$

For every $M > N$ define

$$P(M) = \{z \in \overline{U} : N \leq \text{Re } z \leq M\}.$$

Then, for $j \in \mathbb{Z}$, let $L_j : H \to U$ be defined by the formula

$$L_j(z) = (f|_U)^{-1}(z + 2\pi i j),$$

and let

$$Q_j(M) = L_j(P(M)). \tag{8}$$

The set P(M) and the family of functions

$$\{L_j\}_{j \in \mathcal{K}_M}$$

with

$$\mathcal{K}_M = \{j \in \mathbb{Z} : Q_j(M) \subset \text{Int} P(M)\},$$

define a finite conformal iterated function system . By $J_1(M)$ we denote its limit set. The set $J_1(M)$ is forward $F-$invariant. From (7) and from the fact that the Julia set is the closure of the set of repelling periodic points it follows that

$$J_1(M) \subset J_F. \tag{9}$$

Next we need a condition for j which guarantees that $Q_j(M) \subset \text{Int} P(M)$ (equivalently $j \in \mathcal{K}_M$) for all M large enough. Observe that

$$\mathcal{K}_M \subset \mathcal{K}_{M+1} \tag{10}$$

for all M large enough. To prove (10), let $j \in \mathcal{K}_M$ and let $z \in Q_j(M+1) \setminus Q_j(M)$. Note that, if we assume that $M > M_2 e^{(n-k)(N+1)}$, then we can be sure that Re $z > N+1$ (n and k

are defined in section 2.1.1). Therefore, to get (10), it is enough to prove that Re $z < M+1$. Since

$$F(Q_j(M+1) \setminus Q_j(M)) = P(M+1) \setminus P(M),$$

it follows from Lemma 2.1 that $|F'(z)| \geq \frac{M_1}{M_2}|f(z)| \geq M$ and, then,

$$Q_j(M+1) \setminus Q_j(M) \subset B\Big(z, \frac{M_2 2\pi}{M_1 M}\Big) \subset B(z,1).$$

But we know, that, for $y \in Q_j(M)$, Re $y \leq M$. This proves (10).

The next step is to prove that there exists $j_0 \in \mathbb{N}$ such that, for all $M \in \mathbb{N}$ large enough,

$$j_0, j_0+1, \ldots, e^{\lfloor M/2 \rfloor} \in \mathcal{K}_M. \tag{11}$$

Note that we can find j_0 such that, for every $j \geq j_0$, Re $Q_j(M) > N$. By Lemma 2.1 it is enough to take

$$j_0 = \left\lceil \frac{M_2 e^{(n-k)N} + 2\pi}{\pi} \right\rceil.$$

So, to prove (11) it remains to show that $j < e^{\lfloor M/2 \rfloor}$ implies

$$\text{Re } Q_j(M) \leq M.$$

Striving for a contradiction, suppose that $j < e^{\lfloor M/2 \rfloor}$ and there exists $z \in Q_j(M)$ such that Re $z > M$. Then by Lemma 2.1 we have

$$|f(z)| > M_1 e^{(n-k)M}. \tag{12}$$

Since $z \in Q_j(M)$, $f(z) \in P(M) + 2\pi i j$. Then the square of the distance from zero to the upper-right corner of $P(M) + 2\pi i j$ is greater than $|f(z)|^2$, i.e.

$$M^2 + (s + \pi + 2\pi j)^2 > |f(z)|^2.$$

By (12) and the assumption $j < e^{\lfloor M/2 \rfloor}$, it follows that

$$(M_1 e^{(n-k)M})^2 < M^2 + (s + \pi + 2\pi)^2 e^M.$$

Hence we have the required contradiction since for large M the inequality is false.

Finally observe that by Lemma 2.1, for $j \in \mathcal{K}_M$ and $z \in Q_j(M)$, the following is true

$$|F'(L_j(z+2j\pi i))| \leq \frac{M_2}{M_1}|f(L_j(z+2\pi ij))| \leq \frac{M_2}{M_1}(2j\pi + 2\pi + M).$$

Then

$$P_z(1,1) = \sum_{y \in F^{-1}(z) \cap J_1(M)} \tfrac{1}{|F'(y)|} = \sum_{j \in \mathcal{K}_M} |L_j'(z+2j\pi i)|$$

$$\geq \sum_{j=j_0}^{e^{\lfloor M/2 \rfloor}} \tfrac{1}{\frac{M_2}{M_1}(2j\pi+2\pi+M)}.$$

Since, if M is large enough, the right side of this inequality can be as large as we want, and the proposition are proved. $\square$

APPENDIX

A B C 1 2 3

In this appendix we have a couple of fancyish diagrams and a floating table.

$$\begin{tikzcd}
A \otimes B \arrow[r, "id\otimes p_B"] \arrow[d, "p_A\otimes id"'] & A \otimes \mathbb{Q}_X \arrow[r, "m_2"] \arrow[d, "p_A\otimes id"] & A \arrow[dd, "p_A"] \\
\mathbb{Q}_X \otimes B \arrow[r, "id\otimes p_B"'] \arrow[d, "m_1"'] & \mathbb{Q}_X \otimes \mathbb{Q}_X \arrow[dr, "m"'] & \\
B \arrow[rr, "p_B"'] & & \mathbb{Q}_X
\end{tikzcd}$$

$$\begin{tikzcd}
R\xi_!\xi^!\mathbb{Q}_Y \otimes \mathbb{Q}_Y \arrow[r, "\mathrm{pr}_\xi"] \arrow[dd, "\epsilon_\xi^!\otimes id"'] & R\xi_!(\xi^!\mathbb{Q}_Y \otimes \xi^*\mathbb{Q}_Y) \arrow[d, "R\xi_!(id\otimes\alpha_\xi)"] \\
& R\xi_!(\xi^!\mathbb{Q}_Y \otimes \mathbb{Q}_X) \arrow[d, "\Phi_\xi^{-1}(m_2)"] \\
\mathbb{Q}_Y \otimes \mathbb{Q}_Y \arrow[r, "m"'] & \mathbb{Q}_Y
\end{tikzcd}$$

TABLE A.2. Roots and root vectors for $\mathfrak{so}_{2n+1}$

W	W_I									
E_6	A_2^2	$A_1A_2^2$	A_5							
E_7	$(A_1^3)'$	$A_1^3A_2$	A_5'	$A_1A_2A_3$	A_2A_4	A_1A_5	A_6	A_1D_5	D_6	E_6
E_8	$A_1A_2A_4$	A_3A_4	A_1A_6	A_7	A_2D_5	D_7	A_1E_6	E_7		
F_4	A_2	$\widetilde{A}_2$	C_3	B_3	$A_1\widetilde{A}_2$	$\widetilde{A}_1A_2$				
G_2	A_1	$\widetilde{A}_1$								
H_3	A_1A_1	A_2	$I_2(5)$							
H_4	A_1A_2	A_3	$A_1I_2(5)$	H_3						

Equation and theorem numbering in an appendix will almost certainly be funky.

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