AA 462 Rocket Propulsion

Flight and Staging Lecture

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The material used for this lecture can be found in *Rocket Propulsion Elements* written by George Sutton, Oscar Biblarz, as well as, *Orbital Mechanics for Engineering Students* written by Howard Curtis.

Rocket science is hard. It's almost darn near impossible. To help reduce some of this complexity engineers make assumptions about the system to reduce equations into solvable forms. A common assumption made for rocket propulsion is that the rocket has a constant mass flow as seen in the equation below.

$$\dot{m} = const. = \frac{m_p}{t_p} \tag{1}$$

Where m_p is the total propellant left (i.e. usable propellant) and t_p is the total burn time. Lets use this assumption to examine the velocity change of a rocket while it's in space. Since the rocket at question is assumed to be in space, we will make two more assumptions; the space environment is zero-gravity and there is no drag. Recall from elementary physics, thrust can be expressed as

$$F = m \cdot \frac{du}{dt} \tag{2}$$

The mass is a function of time and is shown in Eq. 3. Here m_0 is the starting initial mass, the mass of the propellant plus the mass of the rocket final rocket structure.

$$m = m_0 - \dot{m}t \tag{3}$$

A substitution can be made for $\dot{m}$ and Eq. 1. The result is Eq. 4.

$$m = m_0 - \frac{m_p}{t_p}t \tag{4}$$

From here we can define two new terms the propellant mass fraction and mass ratio, ζ and MR respectively. The mass fraction, ζ, is defined as the remaining propellant mass divided by the initial mass. By factoring out m_0 from Eq. 4, we can represent the mass as a function of the mass fraction.

$$m = m_0 \left(1 - \zeta \frac{t}{t_p}\right) \tag{5}$$

This can be done for the mass ratio shown in Eq. 6. Note that the mass ratio is the final mass to the initial mass.

$$m = m_0 \left[1 - (1 - MR)\frac{t}{t_p}\right] \tag{6}$$

The relationship between the mass ratio and the mass fraction is

$$\zeta + MR = \frac{m_p}{m_0} + \frac{m_f}{m_0} = \frac{m_p + m_f}{m_0} = \frac{m_0}{m_0} = 1$$

$$\zeta + MR = 1 \tag{7}$$

It was shown in the previous lecture that the characteristic exhaust velocity can be represented as the following

$$c = I_s \cdot g_0 = \frac{F}{\dot{m}} \tag{8}$$

Using Eq. 2 and looking to solve for the velocity. yields the following

$$du = \frac{F}{m} dt$$

Using equations 3, 4, and 8 shown previously, we make several substitutions into the above equation

$$\begin{aligned} du &= \frac{F}{m} dt \\ &= \frac{c\dot{m}}{m} dt \\ &= \frac{c(m_p/t_p)}{m_0 - m_p \frac{t}{t_p}} dt \end{aligned}$$

By factoring out $1/m_0$ from the numerator and denominator yields the equation in terms of the mass fraction

$$du = \frac{c\zeta/t_p}{1 - \zeta t/t_p} dt \tag{9}$$

Integrating the above with boundary conditions u_0 at $t = 0$ and $u = u_p$ at $t = t_p$ yields the following

$$u_p - u_0 = -c\ln(1 - \zeta)$$

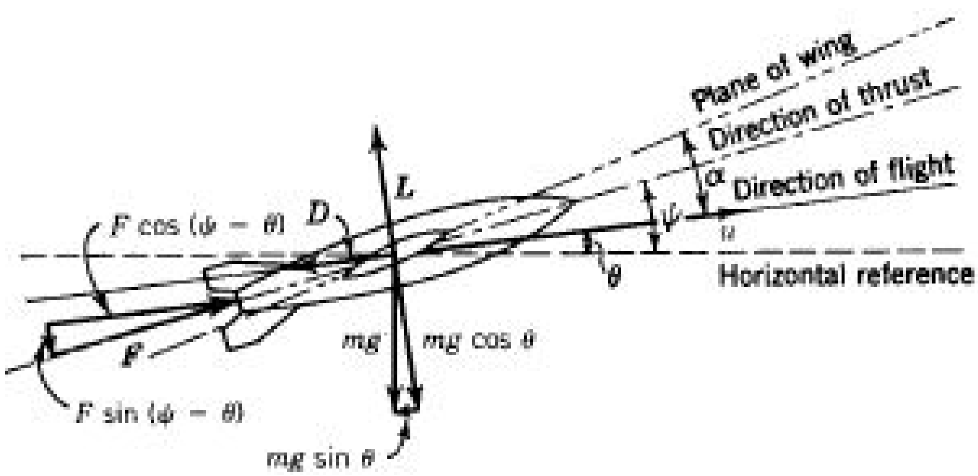
Typically u_0 is assumed to be zero and the velocity u_p is usually designated as the change of velocity Δv. With this knowledge the above equation becomes the following

$$\Delta v = c \ln \frac{1}{1 - \zeta} \tag{10}$$

$$= c \ln \frac{1}{MR} \tag{11}$$

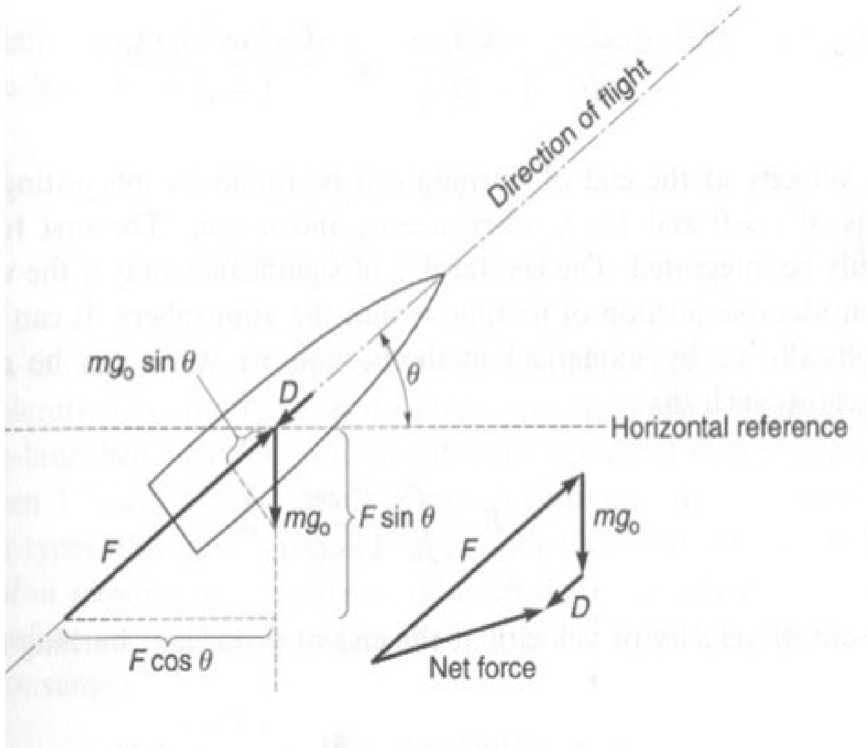
$$= c \ln \frac{m_0}{m_f} \tag{12}$$

Great! We have a simplified equation for the change in velocity for our spacecraft in space. What about when the spacecraft is on it's way to space and is being affected by gravity and atmospheric drag? These are great questions! Thanks for asking. Your welcome. The picture below displays the many forces acting on a spacecraft before it reaches the zero-gravity and zero drag we assumed at the beginning of the lecture.



The forces on a vehicle in flight

To help reduce the complexity of these equations we can make some assumptions about the direction of the thrust and that our rocket won't have lift. This reduced figure can be seen below



Vehicle in flight with assumptions

Examining the sum of the forces in the direction of flight yields

$$m \cdot \frac{du}{dt} = F - mgsin\theta - D$$

Dividing both sides by mass and making Eq. 4 substitution then it will be reduced to the following

$$\frac{dV}{dt} = \frac{C\zeta/t_p}{1-\zeta t/t_p} - g \cdot sin\theta - \frac{\frac{1}{2}C_D \rho V^2 A/m_0}{1-\zeta t/t_p}$$

Integrating both sides for the same boundary conditions used earlier ($t = 0$, $u = u_0$ and $t = t_p$, $u = u_p$).

$$\Delta V = -\bar{C}\ln(1-\zeta) - (\bar{g} \cdot sin\theta) \cdot t_p - \frac{BC_D A}{m_0} \tag{13}$$

Where the average characteristic exhaust velocity and gravity were found. The B in the last term of the equation is the complicated integral below, that can be solved numerically or graphically.

$$B = \int_0^{t_p} \frac{\frac{1}{2}\rho V^2}{1 - \zeta t/t_p} dt \tag{14}$$

Now that we have gone over the flight dynamics of a rocket lets look at the staging of rockets. Some definitions to start us off: where m_f is the final weight of the spacecraft, m_d is the dead weight, and m_0 is the starting mass of the rocket.

Payload fraction: λ

$$\lambda = \frac{m_{pay}}{m_0} = \frac{m_f - m_d}{m_0}$$

Dead weight fraction δ

$$\delta = \frac{m_d}{m_0} = \frac{m_f - m_{pay}}{m_0}$$

Relationships:

$$\frac{m_0}{m_0 - m_p} = \frac{m_0}{m_d + m_{pay}}$$

$$\frac{1}{MR} = \frac{1}{\delta + \lambda}$$

Gravity free space for (N stages) would have us use Eq. 11 and making the substitution from above. Then the change in velocity would be.

$$\Delta V_{tot} = \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} C_i \ln \frac{1}{\delta_i + \lambda_i} = \sum_{i=1}^{N} C_i \ln \frac{1}{1 - \zeta_i}$$

Another helpful equation would be to write the overall payload fraction. This is demonstrated below.

$$\lambda_0 = \frac{m_{pay}}{m_{01}} = \prod_{i=1}^{N} \lambda_i = \frac{m_{pay}}{m_{0N}} \cdot \frac{m_{0N}}{M_{0N-1}} \cdots \frac{m_{03}}{m_{02}} \cdot \frac{m_{f1} - m_{d1}}{m_0 1}$$

The goal now is to maximize the change in velocity for a given set of λ_0, C_i, and δ_i with the restriction that

$$\lambda_0 = \prod_{i=1}^{N} \lambda_i \tag{15}$$

To do this, lets define the function F with Lagrange multiplier K on the payload restriction.

$$F = \sum_{i=1}^{N} C_i \ln \frac{1}{\delta_i + \lambda_i} + K \left[ln \left(\prod_{i=1}^{N} \lambda_i \right) - \ln \lambda_0 \right]$$

Note that natural log was taken to both sides of the restriction then one term was subtracted to the other side. This allows us to incorporate it into the F function. Taking the derivative with respect to λ_i and setting it equal to zero.

$$\frac{\partial F}{\partial \lambda_i} = \frac{-C_i}{\delta_i + \lambda_i} + \frac{K}{\lambda_i} = 0 \qquad \longrightarrow \qquad \lambda_i = \frac{\delta_i}{C_i/K - 1}$$

Plugging this into Eq. 15 yields the following

$$\prod_{i=1}^{N} \lambda_i = \lambda_0 = \frac{\prod\limits_{i=1}^{N} \delta_i}{\prod\limits_{i=1}^{N} \frac{C_i}{K} - 1}$$

The result is an N^{th} order equation for K

$$\prod_{i=1}^{N} \left(\frac{C_i}{K} - 1 \right) = \frac{1}{\lambda_0} \prod_{i=1}^{N} \delta_i \tag{16}$$

Assuming constant C ($C_i = C$) Eq. 16 reduces to

$$\left(\frac{C}{K} - 1 \right)^N = \frac{1}{\lambda_0} \prod_{i=1}^{N} \delta_i = \prod_{i=1}^{N} \left(\frac{\delta_i}{\lambda_i} \right)$$

Then δ_i/λ_i must be constant and can be expressed as

$$\lambda_i = \lambda_0^{1/N} \cdot \frac{\delta_i}{\left(\prod\limits_{i=1}^{N} \delta_i \right)^{1/N}}$$

Adding to the assumptions that the dead weight ratio is constant, $\delta_i = \delta$ then we have the following

$$\lambda_i = \lambda_0^{1/N} \tag{17}$$

From taking the derivative and plugging it back into the Lagrange equation F we arrive to the relationship above. Which demonstrates for a maximum ΔV that all the stages need to have an equal payload ratio λ. Resulting in the final velocity equation

$$\Delta V_{tot} = c \cdot \sum_{i=1}^{N} \ln \frac{1}{\delta + \lambda_i} = c \cdot N \ln \frac{1}{\delta + \lambda_0^{1/N}}$$